How to Construct Integrating Factors

Applications to the Isothermal Parameterization of Surfaces

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Introduction. This short essay has been motivated by a problem encountered in Eisenhart's discussion[1] of the transformation from arbitrary to "isometric" (or "isothermal/isothermic" in a terminology apparently due (∼1837) to Lamé) parameterizations of surfaces in 3-space. I borrow from page 26 of my old *Mathematical Thermodynamics* class notes (1967) and from related material that appears on page 19 , Chapter I of my *Thermal Physics* notes (∼2002).

The construction procedure, and examples. Let

$$\not{d}F = X(x,y)dx + Y(x,y)dy$$

be an inexact differential form: $\partial X/\partial \neq \partial Y/\partial x$. Look to the Pfaffian differential equation

$$\not{d}F = 0$$

which can be written

$$\frac{dy}{dx} = -\frac{X(x,y)}{Y(x,y)} \equiv f(x,y)$$

The equations

$$f(x,y) = c \quad : \quad c \text{ constant}$$

inscribe on the $\{x,y\}$-plane a c-parameterized family of curves. On any such curve we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{dc}{dx} = 0$$

whence

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{X}{Y} = 0$$

giving

$$Y\frac{\partial f}{\partial x} = X\frac{\partial f}{\partial y} \equiv \chi(x,y)XY$$

With χ *thus defined* we have

$$\frac{\partial f}{\partial x} = \chi X \quad \text{and} \quad \frac{\partial f}{\partial y} = \chi Y$$

[1] *Treatise on the Differential Geometry of Curves and Surfaces* (1909), §40.

giving

$$\chi = \frac{1}{X}\frac{\partial f}{\partial x} = \frac{1}{Y}\frac{\partial f}{\partial y}$$

and

$$\chi đF = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

So though $đF$ is inexact, $\chi đF = df$ is an exact differential form—rendered exact by the "integrating factor" $\chi(x,y)$.

By an "n-form we understand expressions of the form

$$đF = \sum_{k=1}^{n} F_k(x^1, x^2, \ldots, x^n)dx^k$$

Every 1-form is manifestly exact, and every 2-form admits (by the preceding construction) of an integrating factor (so is "integrable"). But n-forms with $n \geqslant 3$ admit of integrating factors only exceptionally, as discussed and illustrated in the class notes cited above. We are informed by Ince[2] that, though integrating factors do appear occasionally in earlier literature, the first person to treat the idea in a systematic way was Euler (1734, 1760), and that significant contributions to the subject were made by Clairaut (1739, 1740).

Examples. Look to the 2-form

$$đF \equiv X\,dx + Y\,dy = y\,dx - dy$$

which is transparently inexact: $X_y = 1$ while $Y_x = 0$. Pfaff's differential equation[3] $đF = 0$ can be written

$$\frac{dy}{dx} + \frac{X}{Y} = \frac{dy}{dx} - y = 0$$

The general solution is

$$y(x;a) = e^{x+a} \quad : \quad a \text{ is a constant of integration}$$

which inscribes an a-parameterized population of curves on the $\{x,y\}$-plane. An implicit description of those is provided by

$$f(x,y) = a \quad \text{where} \quad f(x,y) = \log y - x$$

We now have

$$\begin{aligned}\chi &= \frac{1}{X}\frac{\partial f}{\partial x} = -\frac{1}{y}\\ &= \frac{1}{Y}\frac{\partial f}{\partial y} = -\frac{1}{y}\end{aligned}$$

[2] E. L. Ince, *Ordinary Differential Equations* (1926), page 534.

[3] Johann Friedrich Pfaff (1765–1825), a lifelong friend of Friedrich Schiller, studied mathematics at Göttingen and practical astronomy in Berlin under Bode, became the teacher of Gauss—his junior by twelve years— and of Möbius. His brother Johann Wilhelm was also a professor of mathematics, while another brother (Christian Heinrich) was a professor of medicine, physics and chemistry.

which gives

$$\begin{aligned}\chi\, \text{đ}F &= -y^{-1}(y\,dx - dy)\\ &= -dx + y^{-1}dy\\ &= d(-x + \log y) = df\end{aligned}$$

Look now to the (randomly constructed) 2-form

$$\text{đ}F \equiv X\,dx + Y\,dy = x^2y^3dx + x^3y\,dy$$

which is again transparently inexact: $X_y = 3x^2y^2$ while $Y_x = 3x^2y$. From

$$\frac{dy}{dx} + \frac{X}{Y} = \frac{dy}{dx} + \frac{y^2}{x} = 0$$

we obtain

$$y(x;a) = \frac{1}{\log x + a}$$

Implicit description of the associated inscribed curves is provided by

$$f(x,y) = a \quad \text{where now} \quad f(x,y) = y^{-1} - \log x$$

We are led thus to write

$$\begin{aligned}\chi &= \frac{1}{X}\frac{\partial f}{\partial x} = -\frac{1}{x^3y^3}\\ &= \frac{1}{Y}\frac{\partial f}{\partial y} = -\frac{1}{x^3y^3}\end{aligned}$$

which gives

$$\begin{aligned}\chi\text{đ}F &= -\frac{1}{x^3y^3}(x^2y^3dx + x^3y\,dy)\\ &= -\frac{1}{x}dx - \frac{1}{y^2}dy\\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df\end{aligned}$$

Look finally to an example taken from the differential geometry of surfaces, an application of the sort that motivated this discussion. The vector

$$\boldsymbol{r} = \begin{pmatrix}\sin u\cos v\\ \sin u\sin v\\ \cos u\end{pmatrix}$$

provides a standard parametric construction of the unit sphere, and leads to the quadratic form

$$\begin{aligned}ds^2 &= (\boldsymbol{r}_u\cdot\boldsymbol{r}_u)du^2 + 2(\boldsymbol{r}_u\cdot\boldsymbol{r}_v)dudv + (\boldsymbol{r}_v\cdot\boldsymbol{r}_v)dv^2\\ &= du^2 + \sin^2 u\,dv^2\\ &= (du + i\sin u\,dv)(du - i\sin u\,dv)\end{aligned}$$

The factors are complex conjugates of one another. Look to the first:

$$\text{đ}F = U\,du + V\,dv = du + i\sin u\,dv$$

which is again transparently inexact: $0 \neq i\cos u$. The solution of

$$\frac{dv}{du} + \frac{U}{V} = \frac{dv}{du} - i\frac{1}{\sin u} = 0$$

is

$$v(u;a) = i\log\big[2\tan\tfrac{1}{2}u\big] + a$$

Implicit description of the curves thus inscribed on the $\{u,v\}$-plane is provided by

$$f(u,v) = a \quad \text{where} \quad f(u,v) = v - i\log\big[2\tan\tfrac{1}{2}u\big]$$

We are led thus to write

$$\chi = \frac{1}{U}\frac{\partial f}{\partial u} = \frac{1}{V}\frac{\partial f}{\partial v} = -i\csc u$$

and to observe that

$$\chi \,\bar{d}F = -i\csc u\cdot(du + i\sin u\,dv) = (-i\csc u\,du + dv) = df$$

By complex conjugation we have

$$\bar{d}\bar{F} = du - i\sin u\,dv = \frac{1}{\bar{\chi}}d\bar{f}$$

and therefore

$$ds^2 = \bar{d}\bar{F}\bar{d}F = |\bar{d}F|^2 = \frac{1}{|\chi|^2}|df|^2$$

We write

$$\lambda = \frac{1}{|\chi|^2} = \sin^2 u$$

and—since $f(u,v)$ is a function—can write

$$f(u,v) = p(u,v) + iq(u,v)$$

with

$$p(u,v) = v \quad \text{and} \quad q(u,v) = -\log\big[2\tan\tfrac{1}{2}u\big]$$

Inversely,

$$v(p,q) = p \quad \text{and} \quad u(p,q) = 2\arctan\big[\tfrac{1}{2}e^{-q}\big]$$

We have achieved thus a transformation $\{u,v\}\rightarrow\{p,q\}$ from $\{u,v\}$-coordinates to $\{p,q\}$-coordinates, and by means of the latter have brought the metric to a form that is conformally equivalent to the Euclidean metric:

$$ds^2 = \lambda(dp^2 + dq^2)$$

We find

$$\sin u = \sin\big[2\arctan\big[\tfrac{1}{2}e^{-q}\big]\big] = \frac{4e^q}{4e^{2q}+1}$$

$$\cos u = \cos\big[2\arctan\big[\tfrac{1}{2}e^{-q}\big]\big] = \frac{4e^{2q}-1}{4e^{2q}+1}$$

so the sphere has acquired the re-parameterized description

$$\boldsymbol{r} = \begin{pmatrix} \dfrac{4e^q}{4e^{2q}+1}\cos p \\ \dfrac{4e^q}{4e^{2q}+1}\sin p \\ \dfrac{4e^{2q}-1}{4e^{2q}+1} \end{pmatrix}$$

where p ranges on $[0, 2\pi]$ and q ranges on $[-\infty, +\infty]$: the equator occurs at $q = -\frac{1}{2}\log 4 = \log\frac{1}{2}$.

Conformality-preserving transformations. Let $\{p, q\}$ refer (as above) to a conformal parameterization of a surface Σ

$$ds^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = \lambda(p,q)\cdot(dp^2 + dq^2) = \lambda\cdot(dp + idq)(dp - idq)$$

and let them stand in one or the other of the following relationships to new parameters $\{P, Q\}$

$$p + i\,q = Z(P \pm i\,Q) \tag{$\star$}$$

where $Z(\bullet)$ is an arbitrary smooth (real or complex-valued) function of a single variable. Then

$$dp + i\,dq = Z'(P \pm i\,Q)\cdot(dP \pm i\,dQ)$$

gives

$$|dp + i\,dq|^2 = dp^2 + dq^2 = |Z'|^2\cdot(dP^2 + dQ^2)$$

whence

$$ds^2 = \Lambda\cdot(dP^2 + dQ^2)) \quad \text{with} \quad \Lambda(P,Q) = \lambda|Z'|^2$$

We conclude that from any conformal parameterization of Σ can be obtained a double infinity of alternative conformal parameterizations, one for every choice of $Z(\bullet)$. Eisenhart argues[1] that—conversely—every pair of conformal parameters stands in the relation ($\star$).

A trivial example of historic importance. Look to the case

$$p + i\,q = Z(P + i\,Q) = (P + i\,Q)^1 - i\log 2\cdot(P + i\,Q)^0$$

which entails

$$\begin{aligned} p &= P \\ q &= Q - \log 2 = \log\left[\frac{e^Q}{2}\right] \end{aligned}$$

and is found with *Mathematica*'s assistance[4] to send

$$\begin{pmatrix} \dfrac{4e^q}{4e^{2q}+1}\cos p \\ \dfrac{4e^q}{4e^{2q}+1}\sin p \\ \dfrac{4e^{2q}-1}{4e^{2q}+1} \end{pmatrix} \longrightarrow \begin{pmatrix} \operatorname{sech} Q\cos P \\ \operatorname{sech} Q\sin P \\ \tanh Q \end{pmatrix}$$

[4] Use the command `/.q→Log[e^Q/2]//ExpToTrig//Simplify`.

Explicit calculation now supplies a conformal result

$$ds^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = \text{sech}^2 Q \cdot (dP^2 + dQ^2)$$

that conforms to our expectation, since in the present instance $|Z'|^2 = 1$ so[4]

$$\Lambda = \lambda|Z'|^2 = \sin^2 u = \sin^2\!\big(2\arctan\big[\tfrac{1}{2}e^{-q}\big]\big) \longrightarrow \text{sech}^2 Q$$

The example is "trivial" in a sense that emerges from the observation that from ($\star$) we expect generally—by the Cauchy-Riemann equations—to have

$$\frac{\partial p}{\partial P} = \frac{\partial q}{\partial Q} \quad\text{and}\quad \frac{\partial p}{\partial Q} = -\frac{\partial q}{\partial P}$$

which indicate that we should in general expect to have $p = p(P,Q), q = q(P,Q)$: old depend upon both of new (and *vice versa*). But in the present instance the Cauchy-Riemann equations reduce to

$$1 = 1 \quad\text{and}\quad 0 = 0 \quad : \quad \begin{cases} p(P,Q) \text{ linear in } P\text{, independent of } Q \\ q(P,Q) \text{ linear in } Q\text{, independent of } P \end{cases}$$

The example acquires historic importance, however, from the circumstance[5] that it gives rise to the Mercator projection[6] of the terrestrial globe. The projection (onto the cylindar tangent at the equator, which is then unrolled to produce the map) greatly distorts polar regions, but is of nautical importance because—non-obviously, this being an indicator of Mercator's mathematical abilities: he worked a century before the invention of the calculus, and nearly fifty years before the invention (by John Napier, in 1614) of natural logarithms—it renders all "rhumb lines" or "loxodromes" (lines that cross all meridians[7] at the same angle) as straight lines.

Conclusions, so far as concerns the differential geometry of surfaces. In the theory of surfaces Σ in $\mathbb{R}^3$ we benefit from a couple of lucky accidents: (*i*) the quadratic differential form ds^2 can invariably be factored—we have

$$Edu^2 + 2Fdudv + Gdv^2 = \Big(\sqrt{E}du + \frac{F+iH}{\sqrt{E}}dv\Big)\Big(\sqrt{E}du + \frac{F-iH}{\sqrt{E}}dv\Big)$$

where from the Euclidean structure of $\mathbb{R}^3$ it follows that $H \equiv \sqrt{EG - F^2}$ is real—and (*ii*), the factors are 2-forms which invariably are integrable. The metric structure of surfaces Σ in $\mathbb{R}^n$ $(n > 3)$ is described by quadratic forms ds^2 which can be factored only exceptionally, and the factors—when they exist

[5] See Exercise 16 on page 230 of Manfredo do Carmo's *Differential Geometry of Curves and Surfaces* (1976). This excellent text (originally published in Portuguese) is available on the web as a free pdf download. Page 230 appears as panel [238].

[6] Gerardus Mercator (1512–1592) was a Flemish cartographer of great distinction, but accomplished also in remarkably many other areas, as also were his several sons. The Mercator projection appeared in 1569.

[7] Seafarers could use a sextant to determine meridians (lines of constant latitude), but were unable to determine longitude prior to the invention of the marine chronometer by John Harrison (1693–1776) and others. Navigators ignorant of longitude were obliged to assign prime importance to rhomb lines.

—are $(n-1)$-forms, which only exceptionally are integrable. One is deprived, therefore, of the analytical apparatus that would permit one to achieve

$$ds = \sum_{i,j=1}^{n-1} g_{ij}dx^i dx^j = \lambda(u_1, u_2 \ldots, u_{n-1})\cdot(du_1^2 + du_2^2 + \cdots + du_{n-1}^2)$$

In 4-dimensional spacetime the situation would appear, however, to be somewhat more tractable, insofar as surfaces in that context can be considered to be moving surfaces Σ_t in 3-space, so susceptible at each instant to analysis of the type described above.

Incidental remarks. A small adjustment

$$\begin{pmatrix} \operatorname{sech}Q\cos P \\ \operatorname{sech}Q\sin P \\ \tanh Q \end{pmatrix} \longrightarrow \begin{pmatrix} \operatorname{sech}Q\cos P \\ \operatorname{sech}Q\sin P \\ Q-\tanh Q \end{pmatrix}$$

converts (what we may call) Mercator's parameterization of the unit sphere into a standard parameterization of the unit pseudosphere.

An alternative approach to the construction of isothermal coordinate systems —one that I find problematic but that seems to be preferred by most modern authors—was devised by Beltrami (1867). Its success hinges on one's ability to solve the "Beltrami equation," which is of the form

$$\frac{\partial w}{\partial \bar z} = \mu\frac{\partial w}{\partial z}$$

and has generated a vast literature. Gauss (1822), building upon results special to surfaces of revolution that had been obtained by Lagrange (1779), was the first to establish the general existence of isothermal coordinate systems and to appreciate their general utility.[8]

APPENDIX

Isothermal parameterization of the pseudosphere. Restore generic parameters $\{u, v\}$ to the preceding construction of the unit pseudosphere, writing

$$\boldsymbol{r} = \begin{pmatrix} \operatorname{sech}u\cos v \\ \operatorname{sech}u\sin v \\ u-\tanh u \end{pmatrix}$$

Quick calculation supplies

$$\begin{aligned} ds^2 &= \tanh^2 u\, du^2 + \operatorname{sech}^2 v\, dv^2 \\ &= (\tanh u\, du + i\operatorname{sech} v\, dv)(\tanh u\, du - i\operatorname{sech} v\, dv) \end{aligned}$$

Let

$$\begin{aligned} đF &= U(u,v)du + V(u,v)dv \\ &= \tanh u\, du + i\operatorname{sech} v\, dv \quad : \quad \text{manifestly inexact} \end{aligned}$$

[8] See, for example, the Wikipedia articles "Isothermal coordinates" and "Beltrami equation."

Look to the associated Pfaff equation $\text{đ}F = 0$ which can be written

$$\frac{dv}{du} = -\frac{U}{V} = i \sinh u$$

the general solution of which is

$$v(u; a) = i \cosh u + a$$

Define

$$f(u, v) = v - i \cosh u$$

which by $f(u, v) = a$ inscribes an a-parameterized population of curves on the $\{u, v\}$-plane. Construct

$$\chi(u, v) = \frac{1}{U}\frac{\partial f}{\partial u} = \frac{1}{V}\frac{\partial f}{\partial v} = -i \cosh u$$

and obtain

$$\chi \text{đ}F = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df = dv - i \sinh u \, du$$

whence

$$ds^2 = \text{đ}F \cdot \text{đ}\bar{F} = \frac{1}{|\chi|^2}|df|^2$$

Write

$$f(u, v) = p(u, v) - i\, q(u, v) \quad : \quad \begin{cases} p(u, v) = v \\ q(u, v) = \cosh u \end{cases}$$

and obtain the conformal result

$$ds^2 = \lambda(dp^2 + dq^2) \quad \text{with} \quad \lambda = \text{sech}^2 u$$

Which is to say

$$\tanh^2 u \, du^2 + \text{sech}^2 v \, dv^2 = \text{sech}^2 u \cdot (dv - i \sinh u)(dv + i \sinh u)$$

By functional inversion

$$\begin{aligned} u &= \text{arccosh}\, q \\ v &= p \end{aligned}$$

which when introduced into $\boldsymbol{r}(u, v)$ give

$$\boldsymbol{r}(p, q) = \begin{pmatrix} q^{-1} \cos p \\ q^{-1} \sin p \\ \text{arccosh}\, q - q^{-1}\sqrt{q^2 - 1} \end{pmatrix}$$

where p ranges on $[0, 2\pi]$ and q ranges on $[1, \infty]$. As was previously remarked, from this particular isothermal parameterization of the unit pseudosphere infinitely many others could be produced.